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FAST TRACK COMMUNICATION

Tetrahedron equations, boundary states and the hidden structure of $\mathcal{U}_q\left(D_n^{(1)}\right)$

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Abstract
Simple periodic 3D → 2D compactification of the tetrahedron equations gives the Yang–Baxter equations for various evaluation representations of $\mathcal{U}_q(\widehat{sl}_n)$. In this paper we construct an example of fixed non-periodic $3D$ boundary conditions producing a set of Yang–Baxter equations for $\mathcal{U}_q\big(D_n^{(1)}\big)$. These boundary conditions resemble a fusion in the hidden direction.

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The tetrahedron equation can be viewed as a local condition providing existence of an infinite series of Yang–Baxter equations. In the applications to quantum groups the method of the tetrahedron equation is a powerful tool for the generation of R-matrices and L-operators for various 'higher spin' evaluation representations. In the framework of the elder Zamolodchikov–Bazhanov–Baxter and related models and $\mathcal{U}_q(\widehat{sl}_n)$ this has been known for a long time, see e.g. [1, 2, 5, 6]. In the framework of the more novel q-oscillator model it has been demonstrated in [3] for $\mathcal{U}_q(\widehat{sl}_n)$ and in [8] for super-algebras $\mathcal{U}_q(\widehat{gl}_{n|m})$.

The main principle producing the cyclic $\widehat{sl}_n$ structure is the trace of three-dimensional monodromy operators in the hidden 'third' direction. In this paper we introduce another non-periodic boundary condition for the q-oscillator scheme, namely specific fixed boundary states in the hidden direction still providing the existence of effective Yang–Baxter equations with multiplicative spectral parameters. This is a new example of three-dimensional *integrable* boundary conditions producing the spectral decomposition of commutative layer-to-layer transfer matrices.

We shall start with a short reminder of a (super-)tetrahedron equation and $\widehat{sl}_n$ compactification in their elementary form. The simplest known tetrahedron equation in the tensor product of six spaces $B_1 \otimes F_2 \otimes \cdots \otimes F_5 \otimes B_6$ is

$$\mathsf{R}_{B_1F_2F_3}\mathsf{R}_{B_1F_4F_5}\mathsf{R}_{F_2F_4B_6}\mathsf{R}_{F_3F_5B_6} = \mathsf{R}_{F_3F_5B_6}\mathsf{R}_{F_2F_4B_6}\mathsf{R}_{B_1F_4F_5}\mathsf{R}_{B_1F_2F_3}, \tag{1}$$

where $F_i = \{|0\rangle, |1\rangle\}_i$ is a representation space of the Fermi oscillator

$$\boldsymbol{f}^+|0\rangle = |1\rangle, \qquad \boldsymbol{f}^-|1\rangle = |0\rangle. \tag{2}$$

Odd operators $\boldsymbol{f}_i^\pm$ in different components i of their tensor product anti-commute and $\left(\boldsymbol{f}_i^\pm\right)^2 = 0$. It is convenient to introduce projectors

$$\boldsymbol{M}_i = \boldsymbol{f}_i^+\boldsymbol{f}_i^-, \qquad \boldsymbol{M}_i^0 = \boldsymbol{f}_i^-\boldsymbol{f}_i^+, \qquad \left[\boldsymbol{f}_i^+, \boldsymbol{f}_i^-\right]_+ = \boldsymbol{M}_i^0 + \boldsymbol{M}_i = 1. \tag{3}$$

Operator $\boldsymbol{M}_i^0$ is the projector to the vacuum, $\boldsymbol{M}_i$ is the occupation number and $\boldsymbol{M}^0\boldsymbol{M} = 0$.

Space B_i stands for representation space of the ith copy of the q-oscillator,

$$\boldsymbol{b}^+\boldsymbol{b}^- = 1 - q^{2\boldsymbol{N}}, \qquad \boldsymbol{b}^-\boldsymbol{b}^+ = 1 - q^{2\boldsymbol{N}+2}, \qquad q^{\boldsymbol{N}}\boldsymbol{b}^\pm = \boldsymbol{b}^\pm q^{\boldsymbol{N}\pm 1}. \tag{4}$$

In this paper we imply the unitary Fock space representation, $(\boldsymbol{b}^-)^\dagger = \boldsymbol{b}^+$, defined by

$$\boldsymbol{N}|n\rangle = |n\rangle n, \qquad \boldsymbol{b}^-|0\rangle = 0, \qquad |n\rangle = \frac{\boldsymbol{b}^{+n}}{\sqrt{(q^2;q^2)_n}}|0\rangle, \qquad n \geqslant 0, \tag{5}$$

where $(x;q^2)_n = (1-x)(1-q^2x)\cdots(1-q^{2n-2}x)$. In terms of creation, annihilation and occupation number operators the R-matrices in (1) are given [8] by

$$\mathsf{R}_{B_1F_2F_3} = \boldsymbol{M}_2^0\boldsymbol{M}_3^0 - q^{\boldsymbol{N}_1+1}\boldsymbol{M}_2\boldsymbol{M}_3^0 + q^{\boldsymbol{N}_1}\boldsymbol{M}_2^0\boldsymbol{M}_3 - \boldsymbol{M}_2\boldsymbol{M}_3 + \boldsymbol{b}_1^-\boldsymbol{f}_2^+\boldsymbol{f}_3^- - \boldsymbol{b}_1^+\boldsymbol{f}_2^-\boldsymbol{f}_3^+ \tag{6}$$

and

$$\mathsf{R}_{F_1F_2B_3} = \boldsymbol{M}_1^0\boldsymbol{M}_2^0 + \boldsymbol{M}_1\boldsymbol{M}_2^0q^{\boldsymbol{N}_1+1} - \boldsymbol{M}_1^0\boldsymbol{M}_2q^{\boldsymbol{N}_1} - \boldsymbol{M}_2\boldsymbol{M}_3 + \boldsymbol{f}_1^+\boldsymbol{f}_2^-\boldsymbol{b}_3^- - \boldsymbol{f}_1^-\boldsymbol{f}_2^+\boldsymbol{b}_3^+. \tag{7}$$

Both operators R are unitary roots of unity. The constant tetrahedron equation (1) can be verified in the operator language straightforwardly.

Define next the 'monodromy' of R-matrices as the ordered product

$$\mathsf{R}_{\Delta_n(B_1F_2),F_3} = \mathsf{R}_{B_{1:1}F_{2:1}F_3}\mathsf{R}_{B_{1:2}F_{2:2}F_3}\cdots\mathsf{R}_{B_{1:n}F_{2:n}F_3} \leftrightharpoons \prod_{j=1..n}^{\curvearrowright}\mathsf{R}_{B_{1:j}F_{2:j}F_3}. \tag{8}$$

Here the convenient 'co-product' notation stands for a tensor power of corresponding spaces,

$$\Delta_n(B_1) = \bigotimes_{j=1}^n B_{1:j}, \qquad \Delta_n(F_2) = \bigotimes_{j=1}^n F_{2:j}. \tag{9}$$

The repeated use of (1) provides

$$\mathsf{R}_{\Delta_n(B_1F_2),F_3}\mathsf{R}_{\Delta_n(B_1F_4),F_5}\mathsf{R}_{\Delta_n(F_2F_4),B_6}\mathsf{R}_{F_3F_5B_6} = \mathsf{R}_{F_3F_5B_6}\mathsf{R}_{\Delta_n(F_2F_4),B_6}\mathsf{R}_{\Delta_n(B_1F_4),F_5}\mathsf{R}_{\Delta_n(B_1F_2),F_3}. \tag{10}$$

Note the conservation laws

$$v^{-\boldsymbol{M}_3}u^{-\boldsymbol{M}_5}\left(\frac{u}{v}\right)^{\boldsymbol{N}_6}\mathsf{R}_{F_3F_5B_6} = \mathsf{R}_{F_3F_5B_6}v^{-\boldsymbol{M}_3}u^{-\boldsymbol{M}_5}\left(\frac{u}{v}\right)^{\boldsymbol{N}_6}. \tag{11}$$

Multiplying (10) by the u, v-term in $F_3 \otimes F_5 \otimes B_6$ and by $\mathsf{R}_{F_3F_5B_6}^{-1}$, and making then the traces over $F_3 \otimes F_5 \otimes B_6$, we come to the Yang–Baxter equation

$$L_{\Delta_n(B_1F_2)}(v)L_{\Delta_n(B_1F_4)}(u)R_{\Delta_n(F_2F_4)}(u/v) = R_{\Delta_n(F_2F_4)}(u/v)L_{\Delta_n(B_1F_4)}(u)L_{\Delta_n(B_1F_2)}(v), \tag{12}$$

where

$$L_{\Delta_n(B_1F_2)}(v) = \underset{F_3}{\mathrm{Str}}\left(v^{-\boldsymbol{M}_3}\mathsf{R}_{\Delta_n(B_1F_2),F_3}\right), \qquad R_{\Delta_n(F_2F_4)}(w) = \underset{B_6}{\mathrm{Tr}}\left(w^{\boldsymbol{N}_6}\mathsf{R}_{\Delta_n(F_2F_4),B_6}\right). \tag{13}$$

This is the case of $\mathscr{U}_q(\widehat{sl}_n)$. Two-dimensional R-matrices (13) have the centers

$$J_i = \sum_{j=1}^n \boldsymbol{M}_{i:j} \quad \text{for fermions and } J_1 = \sum_{j=1}^n \boldsymbol{N}_{1:j} \quad \text{for bosons.} \tag{14}$$

Irreducible components of R-matrices and L-operators (13) correspond to fixed values of J_i. In particular, $\Delta_n(F)$ is the sum of all antisymmetric tensor representations of sl_n,

$$\dim \Delta_n(F) = 2^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}. \tag{15}$$

The Dirac spinor representation of D_n has the same dimension 2^n, it is the direct sum of two irreducible Weyl spinors with dimensions 2^{n-1}. It is evident intuitively, the structure of D_n will appear if the total occupation number J of $\Delta_n(F)$ is not a center of L-operators and R-matrices, but all operators preserve just the parity of J. Also, since the dimension of the vector representation of D_n is $2n$, we need to double the number of bosons.

Consider now two copies of (1) and further of (10) glued in the 'second' direction. This consideration keeps the desired space $\Delta_n(F)$ and doubles the number of bosons. The repeated use of (1) provides

$$\mathsf{R}_{\Delta(B_1)F_2\Delta(F_3)}\mathsf{R}_{\Delta(B_1)F_4\Delta(F_5)}\mathsf{R}_{F_2F_3B_6}\mathsf{R}_{\Delta'(F_3F_5)B_6} = \mathsf{R}_{\Delta'(F_3F_5)B_6}\mathsf{R}_{F_2F_3B_6}\mathsf{R}_{\Delta(B_1)F_4\Delta(F_5)}\mathsf{R}_{\Delta(B_1)F_2\Delta(F_3)}, \tag{16}$$

where

$$\mathsf{R}_{\Delta(B_1)F_2\Delta(F_3)} = \mathsf{R}_{B_1F_2F_3}\mathsf{R}_{B_1'F_2F_3'} \qquad \text{and} \qquad \mathsf{R}_{\Delta'(F_3F_5)B_6} = \mathsf{R}_{F_3'F_5'B_6}\mathsf{R}_{F_3F_5B_6}. \tag{17}$$

The key observation is the existence of a family of eigenvectors of operator $\mathsf{R}_{\Delta'(F_3F_5)B_6}$:

$$\mathsf{R}_{\Delta'(F_3F_5)B_6}|\psi_{\Delta(F_3)}(v)\psi_{\Delta(F_5)}(u)\psi_{B_6}(u/v)\rangle = \big|\psi_{\Delta(F_3)}(v)\psi_{\Delta(F_5)}(u)\psi_{B_6}(u/v)\big\rangle, \tag{18}$$

where

$$\Delta(F) = F' \otimes F, \qquad |\psi_{\Delta(F)}(v)\rangle = (1 + v^{-1}\boldsymbol{f}^{+\prime}\boldsymbol{f}^{+})|0\rangle, \tag{19}$$

and the state $\psi_B(w)$ satisfies $(\boldsymbol{b}^- - w\boldsymbol{b}^+)|\psi_B(w)\rangle = 0$; in the unitary basis (3) its matrix elements are

$$\langle 2k+1|\psi_B(w)\rangle = 0, \qquad \langle 2k|\psi_B(w)\rangle = w^k\sqrt{\frac{(q^2;q^4)_k}{(q^4;q^4)_k}}. \tag{20}$$

The normalization of ψ_B is given by

$$\langle\overline{\psi}_B(w)|(\boldsymbol{b}^{\pm})^{2m}|\psi_B(w)\rangle = w^m\frac{(q^{2+4m}w^2;q^4)_\infty}{(w^2;q^4)_\infty}. \tag{21}$$

Another property of $|\psi_B\rangle$ is

$$R_{B_1,B_2,B_3}\big|\psi_{B_1}(v)\psi_{B_2}(u)\psi_{B_3}(u/v)\big\rangle = \big|\psi_{B_1}(v)\psi_{B_2}(u)\psi_{B_3}(u/v)\big\rangle. \tag{22}$$

Analytical proof of this formula is rather complicated.

Considering now a length-n chain of (16) in the 'third' direction and applying vectors $\psi_{\Delta(F_3)}(u)$, $\psi_{\Delta(F_5)}(v)$ and $\psi_B(u/v)$, we come to the Yang–Baxter equation

$$\begin{aligned}&L_{\Delta_n(\Delta(B_1)F_2)}(v)L_{\Delta_n(\Delta(B_1)F_4)}(u)R_{\Delta_n(F_2F_4)}(u/v)\\&\qquad = R_{\Delta_n(F_2F_4)}(u/v)L_{\Delta_n(\Delta(B_1)F_4)}(u)L_{\Delta_n(\Delta(B_1)F_2)}(v)\end{aligned} \tag{23}$$

without trace construction

$$L_{\Delta_n(\Delta(B_1)F_2)}(v) = \big\langle\overline{\psi}_{\Delta(F_3)}(v)\big|\mathsf{R}_{\Delta_n(\Delta(B_1)F_2),\Delta(F_3)}\big|\psi_{\Delta(F_3)}(v)\big\rangle \tag{24}$$

and

$$R_{\Delta_n(F_2F_4)}(w) = \big\langle\overline{\psi}_{B_6}(w)\big|\mathsf{R}_{\Delta_n(F_2F_4),B_6}\big|\psi_{B_6}(w)\big\rangle. \tag{25}$$

Matrix elements of $R_{\Delta_n(F_2F_4)}(w)$ can be calculated with the help of (21) and similar identities. The invariants of L-operator (24) and R-matrix (25) are the parity of $J_2 = \sum \boldsymbol{M}_{2:j}$, similar parity of J_4 and

$$J_1 = \sum_{j=1}^{n} (\boldsymbol{N}_{1:j} - \boldsymbol{N}'_{1:j}). \tag{26}$$

A choice of different spectral parameters in bra- and ket-vectors in (24) and (25) is equivalent to the choice of equal spectral parameters by means of a gauge transformation.

The structure of D_n representation ring can be verified explicitly by a direct calculation of matrix elements of R-matrix (25) for small n and check of factor powers of $\det(\lambda - R)$.

As to $2n$-bosons space, irreducible components of $\Delta_n(\Delta(B_1))$ are in general infinite dimensional. However, a choice of Fock and anti-Fock space representations, $\text{Spectrum}(\boldsymbol{N}_{1:j}) = 0, 1, 2, \ldots$ and $\text{Spectrum}(\boldsymbol{N}'_{1:j}) = -1, -2, -3, \ldots$, makes $\Delta_n(\Delta(B_1))$ a direct sum of symmetric tensors of $O(2n)$.

The main result of this paper is a step forward to a classification of *integrable boundary conditions* in three-dimensional models. At least two scenarios are hitherto known: the quasi-periodic boundary condition (13) and the boundary states condition (24) and (25). These conditions can be imposed for a layer-to-layer transfer matrix in different directions independently. In both scenarios the spectral parameters of effective two-dimensional models reside in the boundary. Also, the boundary admits twists making the quantum groups classification inapplicable [7]. It is worth noting one more possible scenario of integrable boundary conditions: yet unknown $3D$ reflection operators satisfying the tetrahedron reflection equations [4].

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